

Strong Coupling Singularities and Non-Abelian Gauge Symmetries in $N = 2$ String Theory

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We study a class of extremal transitions between topological distinct Calabi–Yau manifolds which have an interpretation in terms of the special massless states of a type II string compactification. In those cases where a dual heterotic description exists the exceptional massless states are due to genuine strong (string-) coupling effects. A new feature is the appearance of enhanced non-abelian gauge symmetries in the exact nonperturbative theory.

1. Introduction

The understanding of the nonperturbative dynamics of $N = 2$ Yang–Mills theories in four dimensions [1,2] has subsequently led to a substantial improvement of the understanding of nonperturbative effects in $N = 2$ string theories. A striking example is the physical realization of transitions between topological different Calabi–Yau manifolds through black hole condensation in the corresponding type II string compactification [3,4]. A second highlight is the conjecture of a $N = 2$ type II – heterotic string duality [5,6], which allows to extract the exact nonperturbative vector moduli space of the heterotic theory from tree-level data of a related type II string [7]. A crucial insight is the interpretation of certain singularities in the exact effective theory as the effect of generically massive solitonic BPS-states, which become massless at certain points in the moduli space. While the effective theory with generically massive states integrated out diverges when approaching these points, a physical sensible theory can be obtained by including the additional massless states as new degrees of freedom.

The conifold singularities [8] are well understood, both on the Calabi–Yau side as well as on the heterotic side, however they represent only one of many types of possible singularities of Calabi–Yau spaces [9]. A first example of these transitions through more general singularities arises very natural as the “mirror transition” to the standard example of a conifold transition from the quintic in \mathbb{P}^4 ($\mathbb{P}^4 \parallel 5$) $_{-200}^{1,101}$ to the complete intersection Calabi–Yau manifold $\left(\begin{array}{c} \mathbb{P}^4 \\ \mathbb{P}^1 \end{array} \parallel \begin{array}{cc} 4 & 1 \\ 1 & 1 \end{array} \right)_{-168}^{2,86}$. This conifold transition was interpreted in [4] in the type IIB picture¹, where the vector moduli space correspond to the complex structure deformations (and hence the $(2,1)$ -forms) and decreases from 101 to 86, by the Higgs-breaking mechanism through massless solitonic BPS hyper multiplets, which come from three-branes wrapping around the vanishing three-cycles. The inverse process in the type IIA picture, where the vector multiplets correspond to Kähler structure deformations (and hence $(1,1)$ -forms) and their number decreases from two to one, involves a new type of transition, which we will study in detail. It has some similarities with a generic type of singularities, which appear in the duals of heterotic theories and are not continuously connected to the weak coupling regime². It is the aim of this paper to understand the

¹ Our conventions are that we consider a type IIA theory compactified on the Calabi–Yau manifold X and a corresponding type IIB theory on the mirror manifold \hat{X} .

² Transitions which occur in the perturbative regime of the heterotic string connect exclusively K_3 -fibrations [10] and will be studied in [11]

physics of these two types of singularities, which have in common the fact that the singular locus in the moduli space is described by a simple discriminant factor involving only a certain subset of complex structure moduli z_s^i , e.g. for $i = 1$:

$$\Delta_s = 1 - 4z_s, \quad (1.1)$$

where z_s is a standard toric coordinate in the complex structure moduli space.

As an example consider the two moduli hypersurface Calabi–Yau $(\mathbb{P}_{11226} \parallel 12)_{-256}^{2,128}$ discussed first in [12,13]. Its mirror \hat{X} is described by the degree twelve polynomial

$$\hat{p} = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 - 12\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^6 x_2^6. \quad (1.2)$$

The discriminant locus, where the three-point functions diverge, is described by

$$\Delta = \Delta_c \times \Delta_s = ((1 - z_1)^2 - z_1^2 z_s) \times (1 - 4z_s), \quad (1.3)$$

where

$$z_1 = -\frac{1}{864} \frac{\phi}{\psi^6}, \quad z_s = \frac{1}{4\phi^2}. \quad (1.4)$$

There is substantial evidence by now [5,14,15,7] for the conjecture that the type IIA string theory compactified on X is the dual description of a heterotic string compactified on $K3 \times T^2$. According to the identification of [5], the T and S fields of the heterotic side should be identified (in the large S /weak coupling regime) with the special coordinates corresponding to z_1 and z_s , respectively. In particular, for large S one has:

$$z_1 = \frac{1728}{j(t_1)} + \dots = \frac{1728}{j(T)} + \dots, \quad z_s = q_s f(q_1) + \dots = \exp(-S) + \dots, \quad (1.5)$$

where t_s, t_1 are the special coordinates in the large complex structure limit and $q_k = \exp(2\pi i t_k)$. The discriminant factor Δ_c describes a conifold singularity tangent to the weak coupling divisor $z_s = 0$ in the point $z_1 = 1 \Leftrightarrow T = i$, where the heterotic theory has a perturbative gauge symmetry enhancement $U(1) \rightarrow SU(2)$. In the exact quantum corrected theory this $SU(2)$ is again broken to $U(1)$, as in the global supersymmetric theory. In fact it was shown in [7], how the exact nonperturbative string theory reduces to the Seiberg–Witten result in the point particle limit. The second factor Δ_s is not connected to the weakly coupled regime and a genuine strong coupling singularity. Moreover its local structure is quite different from the conifold case and of the type we will consider in the following.

2. Gravitational index

From the physical point of view we want to interpret the singularity in the vector moduli space as the effect of additional massless particles. In [16] it was shown how the net number of additional massless multiplets, irrespectively of their gauge quantum numbers, is determined from the asymptotic behaviour of the topological amplitude F_1 defined in [17]. If $V(z)$ denotes the period which vanishes at a singular codimension one locus in the moduli space, a leading behaviour

$$F_1 = -\frac{b}{6} \log V \bar{V} \quad (2.1)$$

corresponds to a net contribution of $6b = n_V - n_H$ additional massless supermultiplets in the effective field theory (in the following we denote b as the gravitational index). Remarkably in all examples of conifold singularities considered so far [12][18] the value of b equals one, supporting the picture developed in [3].

Let us explain how the topological limit of F_1 can be calculated for the type II string on a given Calabi-Yau manifold with h deformation parameters. A general expression was given in canonical coordinates $t_i = B + iR_i^2$ near the large Kähler structure (radii) limit ($R_i \rightarrow \infty, z_i \rightarrow 0$) in [17]

$$F_1^{top} = \log \left[\left(\frac{1}{w_0(z_i(t_k))} \right)^{3+h-\frac{\chi}{12}} \det \left(\frac{\partial z_1 \dots \partial z_h}{\partial t_1 \dots \partial t_h} \right) f(z_i(t_k)) \right], \quad (2.2)$$

where w_0 is the locally ($z_i = 0$) unique holomorphic period and the special coordinates t_i are defined by the mirror map as ratios of the geometrical periods as functions of the z_i ; specifically the single logarithmic periods and w_0 , i.e. $t_i = \frac{w_0 \log(z_i) + \sigma_i}{w_0}$, where σ_i is a series in the z_j . The ansatz for the meromorphic function $f(z) = \prod_{i=0}^m \Delta_i^{r_i} \prod_{j=1}^h z_i^{s_i}$ of the moduli has now to be fixed to yield the asymptotic behavior of F_1^{top} at codimension one loci where the Picard-Fuchs system has singularities. The exponents s_i can be determined from the large radius limits $R_i \rightarrow \infty \forall i = 1 \dots h$ of F_1^{top} and the r_i from its interpretation in terms of worldsheet instanton contributions [17]:

$$F_1^{top} = -\frac{2\pi i}{12} \sum_{i=1}^h t_i \int c_2 \wedge J_i - \sum_{n_i} \left[2n_{d_1, \dots, d_h}^{(1)} \log(\eta(\prod_{i=1}^h q_i^{d_i})) + \frac{1}{6} n_{d_1, \dots, d_h}^r \log(1 - \prod_{i=1}^h q_i^{d_i}) \right] + c. \quad (2.3)$$

As the invariants n_{d_1, \dots, d_h}^r for the rational instantons are known from the expansion of the Yukawa couplings, the vanishing (or the knowledge) of elliptic instantons contributions $n_{d_1, \dots, d_h}^{(1)}$ at a given multidegree d_1, \dots, d_h gives further linear equations for the r_i .

In the following we give examples³ of extremal transitions and calculate the gravitational index. In the simplest examples a two Kähler moduli Calabi-Yau space flows to a one Kähler moduli Calabi-Yau space. The transition occurs, when the Kähler modulus of the complexified volume t_s shrinks to zero, as will be shown in the next section; this happens in a codimension one locus in the complexified Kähler moduli space. As $q_s \rightarrow 1$, independently of q_1 the Gromov-Witten invariants $n_k^{(g)}$ for the holomorphic embedding of curves of all genera g and degree k in the one moduli Calabi-Yau are given by $n_k^{(g)} = \sum_s n_{k,s}^{(g)}$, where the $n_{k,s}^{(g)}$ count the number of curves in the two moduli Calabi-Yau manifold. The sum over s turns out to be finite. There are $n_{0,1}^{(0)}$ isolated rational curves which get contracted. According to the interpretation of [21] there will be BPS saturated solitonic states, which arise from two-brane solutions wrapping around the contracting curves and become massless for zero volume.

The physical picture of the transition as a Higgs effect, described below, suggest, that the transition requires a tuning of the complex structure parameters to a special value too. There is no factorization of the two kinds of moduli spaces, because charged fields become massless. In the case of K_3 -fibrations [14] with the identification of the dilaton as suggested in [5], the transition correspond in the heterotic string picture to a genuine strong coupling effect, which occurs as the base \mathbb{P}^1 is shrinking to zero.

N_0	Extremal Transition	$j : \quad r_j \quad b_j$	$i : \quad s_i \quad K_{11i} \quad c_i^2 \quad n_{i,0}^{(0)} \quad n_{0,i}^{(0)}$	$n_{i,j}^{(1)}$
1	$(\mathbb{P}_{1,1,2,2,2}^4 \parallel 8)_{-168}^{2,86}$ \downarrow $(\mathbb{P}^5 \parallel 4, 2)_{-176}^{1,89}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{5}{6} \quad -\frac{2 \cdot 2}{6}$	$1 : -\frac{17}{6} \quad 8 \quad 56 \quad 640 \quad 4$ $2 : -3 \quad 4 \quad 24 \quad 10032$ $1 : \quad \quad 8 \quad 56$	$n_{i,j}^{(1)} = 0,$ $i < 3$
2	$\left(\mathbb{P}^4 \parallel \begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix} \right)_{-168}^{2,86}$ \downarrow $(\mathbb{P}^4 \parallel 5)_{-200}^{1,101}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : \quad \quad -\frac{16}{6}$	$1 : -\frac{31}{6} \quad 5 \quad 50 \quad 640 \quad 16$ $2 : -3 \quad 4 \quad 24 \quad 10032$ $1 : \quad \quad 5 \quad 50$	$n_{i,j}^{(1)} = 0,$ $i < 3$
3	$\left(\mathbb{P}^4 \parallel \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \right)_{-168}^{2,86}$ \downarrow $(\mathbb{P}_{1^4,4}^4 \parallel 8)_{-296}^{1,149}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : \quad \quad -\frac{2 \cdot 32}{6}$	$1 : -\frac{28}{6} \quad 2 \quad 44 \quad 640 \quad 64$ $2 : -3 \quad 4 \quad 24 \quad 10032$ $1 : \quad \quad 2 \quad 44$	$n_{i,j}^{(1)} = 0,$ $i < 3$

³ More examples can be found in [13,18,19,20].

Table 1: *Extremal transitions from K_3 -fibrations Calabi-Yau spaces via the strong coupling singularity. After the indication of the pair of models we list the exponents r_i of F_1^{top} (2.2) at the conifold r_c and the strong coupling singularity r_s as well as the corresponding coefficient of the β -function $b_i = (n_V - n_H)/6$. In the next subdivision the exponent s_i as well the topological three point couplings K_{ijk} , the evaluation of the second Chern class on the i 'th $(1,1)$ -form $c_i^2 = \int_M c_2 \wedge J_i$ and the numbers of rational curves with small bi-degree are given. The last column indicates the vanishing of elliptic curves, which was used to fix the r_i .*

Note that all the examples in table 1 have the same Hodge-numbers and the same modular properties, governed by $\Gamma_0(2)_+$ [14], at the weak coupling limit in a potential dual heterotic description $z_s \rightarrow 0$.⁴ Their strong coupling behaviour is however very different as they exhibit transitions to different one Kähler moduli cases.

In the first model of table 1 as well as in the cases 4-9 in the tables 2-4 below, the discriminant of the Picard-Fuchs equation exhibits, beside $z_i = z_s = 0$ two (three) components corresponding to the conifold(s) $\Delta_c = 0$ and the strong coupling singularity $\Delta_s = (1 - 4z_s) = 0$. Their exponents are fixed from the vanishing of the elliptic curves. However in these cases we have an universal contribution to the asymptotic behavior of F_1^{top} at Δ_s from the Jacobian, which is $\sim \Delta_s^{1/2}$ (see below). Hence the coefficient b_s is given by $b_s = (r_s + \frac{1}{2}) \cdot 2$, where the factor 2 arises from the normalization of the β -function for $SU(2)$.

The second model has only one discriminant component.

$$\Delta_c = (1 - z_s)^5 + (1 - 256z_1)^2 + z_s z_1 (27z_s^3 - 144z_s^2 + 320z_s - 2816) - 1.$$

However one observes that $\Delta_s = (z_s - 1) = (q_s - 1) + (q_s - 1)f(q_s, q_1)$, vanishes again at $t_s = 0$, i.e. as before a \mathbb{P}^1 shrinks to zero size. Furthermore since $w_0(z) \propto \Delta_s^{-1}$ the model has a singular gauge choice. $\Delta_c \propto \Delta_s^5$ and $z_1 \propto \Delta_s^5$ reflects the fact that $z_1 = 0$ and $\Delta_c = 0$ have a point of tangency at $t_s = 0$, which has to be resolved. Adding finally the contribution

⁴ The present weak coupling calculations in the $N = 2$ heterotic string compactifications in fact do not distinguish between them. This ambiguity will be discussed in detail in [11].

from the Jacobian $\det\left(\frac{\partial z}{\partial t}\right) \propto \Delta_s^5$ leads to $a^5 b_s = (3 + 2 - (-\frac{168}{12})) + 5 - \frac{1}{6} \cdot 5 - \frac{31}{6} \cdot 5 = -\frac{16}{6}$, which gives precisely the difference $n_V - n_H$, which correspond to the inverse of the conifold transition described in [4]. This is consistent with the interpretation of $n_{0,1}^{(0)} = 16$ massless hyper multiplets coming from two-branes wrapping around the corresponding isolated curves. From the expansion of Δ_s one sees that there is no Weyl reflection in the monodromy around Δ_s . The transition is in this respect different from the others in table 1-3.

N_0	Extremal Transition	$j : \quad r_j \quad b_j$	$i : \quad s_i \quad K_{11i} \quad c_i^2$	$n_{i,0}^{(0)} \quad n_{0,i}^{(0)}$	$n_{i,j}^{(1)}$
4	$(\mathbb{P}_{1,1,2,2,6}^4 \parallel 12)_{-252}^{2,128}$ \downarrow $(\mathbb{P}_{1^5,3}^5 \parallel 6, 2)_{-256}^{1,129}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{4}{6} \quad -\frac{2 \cdot 1}{6}$	$1 : -\frac{32}{6} \quad 4 \quad 52$ $2 : -3 \quad 2 \quad 24$ $1 : \quad 4 \quad 52$	2496 223752	$n_{i,j}^{(1)} = 0,$ $i < 3$
5	$(\mathbb{P}_{1^2,2^4}^5 \parallel 6, 4)_{-132}^{2,68}$ \downarrow $(\mathbb{P}^6 \parallel 3, 2, 2)_{-144}^{1,73}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{6}{6} \quad -\frac{2 \cdot 3}{6}$	$1 : -\frac{36}{6} \quad 12 \quad 60$ $2 : -3 \quad 6 \quad 24$ $1 : \quad 12 \quad 60$	360 2682	$n_{i,j}^{(1)} = 0,$ $i < 2$
6	$(\mathbb{P}_{1^2,2^5}^6 \parallel 4, 4, 4)_{-112}^{2,58}$ \downarrow $(\mathbb{P}^7 \parallel 2, 2, 2, 2)_{-128}^{1,65}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{7}{6} \quad -\frac{2 \cdot 4}{6}$	$1 : -\frac{19}{6} \quad 16 \quad 64$ $2 : -3 \quad 6 \quad 24$ $1 : \quad 16 \quad 64$	256 1248	$n_{i,j}^{(1)} = 0,$ $i < 4$

Table 2: Extremal transitions starting from K_3 -fibrations at strong coupling. The modular groups at $z_s \rightarrow 0$ are $SL(2, \mathbb{Z})$, $\Gamma_0(3)_+$ and $\Gamma_0(4)_+$ respectively [14].

N_0	Extremal Transition	$j : \quad r_j \quad b_j$	$i : \quad s_i \quad K_{11i} \quad c_i^2$	$n_{i,0}^{(0)} \quad n_{0,i}^{(0)}$	$n_{i,j}^{(1)}$
7	$(\mathbb{P}_{1,2,2,3,4}^4 \parallel 12)_{-144}^{2,74}$ \downarrow $(\mathbb{P}_{1^3,2^2,3}^5 \parallel 6, 4)_{-156}^{1,79}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{6}{6} \quad -\frac{2 \cdot 3}{6}$	$1 : -\frac{22}{6} \quad 2 \quad 32$ $2 : -\frac{9}{2} \quad 3 \quad 42$ $1 : \quad 2 \quad 32$	252 -9252	$n_{i,j}^{(1)} = 0,$ $i < 2$
8	$(\mathbb{P}_{1,2,2,2,7}^4 \parallel 14)_{-240}^{2,122}$ \downarrow $(\mathbb{P}_{1,1,1,1,4}^4 \parallel 8)_{-296}^{1,149}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{17}{6} \quad -\frac{2 \cdot 14}{6}$	$1 : -\frac{28}{6} \quad 2 \quad 44$ $2 : -\frac{23}{2} \quad 7 \quad 126$ $1 : \quad 2 \quad 44$	3 -6	$n_{i,j}^{(1)} = 0,$ $i < 2$

⁵ Technically the situation is very similar as in the birational equivalent representation of the Calabi-Yau N_0 8 by the non Fermat hypersurface $(\mathbb{P}_{1,1,1,1,3}^4 \parallel 7)_{-240}^{2,122}$ discussed in [19]. The second representation has also no explicit discriminant factor Δ_s in the unresolved moduli space, but a singular gauge w_0 at $t_s = 0$ and a proportionality of z_1 , Δ_c and the Jacobian to Δ_s which add up to $\beta_s = -\frac{2 \cdot 14}{6}$. As similar consideration leads to $b_s = -\frac{2 \cdot 32}{6}$ in case N_0 3.

Table 3: *Extremal transitions from Calabi-Yau manifolds with a Z_2 singular curve, which are not K_3 fibrations⁶. The inverse conifold transition to the extremal transition N_0 8 was studied in [19].*

Let us come now to the physical interpretation of the spectrum derived from the gravitational index. Apart from the factor of two, the index agrees with the difference in the Hodge numbers of the pairs of manifolds and fits well the interpretation of additional massless hyper multiplets coupled to the $U(1)$ vector multiplets. Apart from the cases where $b = 0$, there is a sufficient number of hyper multiplets to allow flat directions of the $N = 2$ scalar potential. Giving a vev to scalars in the hyper multiplets one moves onto a new Higgs branch where the $U(1)$ is broken and the moduli space is that of the Calabi-Yau with the corresponding, lower number of vector multiplets. However we will argue that this is only half of the story: in fact the gravitational index is sensitive only to the net number of vector and hyper multiplets due to their equal contributions with opposite sign [16]. Thus the above results fit equally well the situation where on top of the massless hyper multiplets one has additional vector multiplets which enlarge the $U(1)$ factors to a non-abelian gauge group G , accompanied by a hyper multiplet in the adjoint representation of G . Our arguments rely on the discrete symmetries in the local monodromies, which fit the Weyl group of G , and the fact that the transition are related to curve singularities on the Calabi-Yau manifold. A connection between curve singularities and enhanced gauge symmetries has been discussed in [16] in the local context. A similar effect in an exceptional $N = 2$ compactification with a $N = 4$ characteristic spectrum has been investigated in [9].

Let us point out some features of the proposed physics of the singularities. The spectrum of a vector and hyper multiplet in the adjoint representation plus additional matter has positive beta-function coefficient and a non-abelian gauge symmetry is not expected to be broken as in the asymptotic free case [1]. Due to the presence of the adjoint hyper multiplet there are two possible Higgs breaking mechanisms to the Cartan subalgebra (CSA). The breaking through scalars of the CSA in the vector multiplets corresponds to the motion in the complex structure moduli space of the mirror \hat{X} . However to obtain gauge symmetry enhancement we have also to tune to zero the values of the scalars in the hyper multiplets, which belong to the Kähler moduli and can not be represented by polynomial deformations of \hat{p} . Therefore the factorization of Kähler and complex structure moduli spaces fails due to the gauge couplings between the vector and hyper multiplets.

⁶ For case 7.) $K_{122} = K_{222} = 3$, for case 8.) $K_{122} = 21$, $K_{222} = 63$.

3. Curve singularities

It turns out that one type of singularities we want to understand is related to curve singularities in the Calabi–Yau manifold. In particular such singularities can arise if a Calabi–Yau is described as the subvariety of a singular ambient space, as happens to be the case in the example (1.2). The weighted projective space has quotient singularities, if subsets of the weights have a non-trivial factor n in common; according to whether the Calabi–Yau variety intersects the singular set of the ambient space in a point or a curve, its singularities are locally of the type $\mathbf{C}^3/\mathbb{Z}_n$ or $\mathbf{C}^2/\mathbb{Z}_n$, respectively.

The resolution of the ambient space quotient singularity is standard and can be described in terms of toric geometry; we refer to the literature for details [25]. In toric language the singular curve C corresponds to a one-dimensional edge of the dual polyhedron with integral lattice points on it. The resolution process adds a new vertex for each of these points and to each of these vertices corresponds an exceptional \mathbb{P}^1 bundle over C in the blown up of the Calabi–Yau manifold. For a \mathbb{Z}_n curve singularity the intersection matrix of the exceptional \mathbb{P}^1 's is proportional to the Cartan matrix of A_{n-1} with self-intersections normalized to -1 .

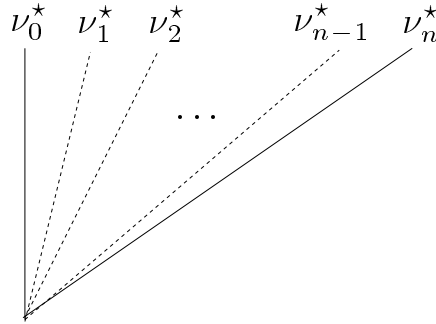


Fig.2 The toric resolution of the \mathbb{Z}_n singularity. To each vertex ν_i^* corresponds a perturbation $X_1^{n-i}X_2^i$ in the defining polynomial.

In virtue of the monomial-divisor map the new vertices correspond to the addition of new perturbations to the original defining polynomial of \hat{X} . The fact that the integral points are arranged on a one-dimensional line translates to the following form of the perturbations:

$$X_1^n, X_1^{n-1}X_2^1, \dots, X_2^n, \quad (3.1)$$

where X_1 and X_2 denote monomials in those projective coordinates, which are zero on the singular curve and the first and last perturbations correspond to the original vertices ν_0^\star and ν_n^\star , respectively. The set of perturbations (3.1) will lead to singular configurations of the mirror Calabi–Yau \hat{X} for special values of the moduli $\phi_0 \dots \phi_N$ of the new perturbations. In particular there is a point in the moduli space where

$$\sum_k \phi_k x^{n-k} y^k = (x + y)^n . \quad (3.2)$$

It is important to note that the singularities of the defining polynomial \hat{p} appear in the complex structure moduli space, whereas the volume of the exceptional divisors of the blow up is controlled by Kähler moduli, which are not described by polynomial deformations of \hat{X} . Thus the relation to the original quotient singularity, leading to the singular curve on X , is not immediately obvious. However it happens that the singular locus in the complex structure moduli space of \hat{X} maps by the mirror map to regions in the Kähler moduli space of X , corresponding to one or more of the exceptional divisors being blown down to a point.

Let us investigate in more detail the simplest quotient singularity \mathbb{Z}_2 . This is the case appearing in the Calabi–Yau manifold (1.2), which we will use as a representative example, although the arguments apply more generally. The locus of the quotient singularity is $x_1 = x_2 = 0$, which is a fixed point of the projective action $x_i \rightarrow \lambda^{w_i} x_i$ with $\lambda = -1$. On \hat{X} there is a singular curve described by

$$x_3^6 + x_4^6 + x_5^2 = 0 . \quad (3.3)$$

The resolution process adds a new vertex ν_1^\star to the dual simplex, which is the average of two of the generic vertices ν_0^\star, ν_2^\star of the singular ambient space,

$$\nu_1^\star = \frac{1}{2}(\nu_0^\star + \nu_2^\star) \quad (3.4)$$

In the polynomial \hat{p} of (1.2), the vertices ν_0^\star, ν_2^\star are related by the monomial-divisor mirror map to the polynomials x_1^{12}, x_2^{12} , while the blow up perturbation related to ν_1^\star is represented by $x_1^6 x_2^6$. Obviously \hat{X} becomes singular for the special value $\phi = \pm 1$ where \hat{p} contains the perfect square $(x_1^6 \pm x_2^6)$; this singularity is the origin of the discriminant factor Δ_s .

The \mathbb{Z}_2 singularity of the ambient space results in the singular locus $\phi^2 = 1$ in the complex structure moduli space of the mirror manifold \hat{X} . More interestingly, the mirror

map related to the complex structure modulus of the blow up, z_s , has the property observed in [14]:

$$0 = \Delta_s = z_s(t_1, t_s) - \frac{1}{4} \Leftrightarrow t_s = 0, \quad (3.5)$$

independently of the value of t_1 . Therefore $\Delta_s = 0$ maps on the type IIA side to a face of the Kähler cone with the corresponding b field set to zero. The resolution of the \mathbb{Z}_2 singularity in the context of transitions through phase boundaries of the complexified Kähler moduli space has been discussed in [26]. The special coordinate t_s is used to define the area of Riemannian surfaces which lie in the exceptional divisor E_s . That is the image of $\Delta_s = 0$ on the type IIB side corresponds to the blow down of E_s on X and the Calabi–Yau develops the original curve singularity. A relation between curve singularities and non-perturbative enhanced gauge symmetries has been established in [27] in a local analysis.

The behavior (3.5) of the mirror map can be traced back to two basic properties of the toric data of the Calabi–Yau variety. The first are those generators of the Mori cone, which express the relations between the vertices arising from the blow up. For $n = 2$ we have a single linear relation, (3.4), which is represented by a vector l_s ,

$$l_s = (1, 1, -2, 0, \dots, 0) . \quad (3.6)$$

In [26] it was shown, how l determines the ordinary hypergeometric differential operator $\mathcal{L}(f) = \theta_z^2 f - z\theta_z(\theta_z + 1/2)f$, ($\theta \equiv z d/dz$), which governs the mirror map between t_s and z_s on a special rational curve, parametrized by z_s , in the compactified moduli space. In a Calabi–Yau phase the rational curve is defined by setting the remaining complex structure moduli to the large complex structure point, $z_i = 0$. The relevant solution of the differential equation is [26]

$$t_s = \frac{1}{2\pi i} \ln \left(\frac{1 - 2z_s - 2\sqrt{1 - 4z_s}}{2z_s} \right) . \quad (3.7)$$

First note that t_s vanishes for $z_s = \frac{1}{4}$. Secondly, under the motion of z once around $z = \frac{1}{4}$, t_s transforms as $t_s \rightarrow -t_s$. The period t_s is a special coordinate of the large complex structure limit; in fact it will correspond to the heterotic dilaton in the cases where the dual description exist. A particle charged only w.r.t. the corresponding $U(1)$ factor becomes massless at $t_s = 0$; moreover it is interpreted as an *electrically* charged excitation w.r.t. the basis chosen in the large complex point (possibly associated to a weakly coupled heterotic theory).

To extend our arguments from $z_i = 0$ to the whole codimension one locus $\Delta_s = 0$, assume that the discriminant factor of the CY contains the factor $\Delta_s = z_s - \frac{1}{4}$. It is clear the the local monodromy around $\Delta_s = 0$ and thus also the leading behavior of the periods in an expansion around $\epsilon = z - \frac{1}{4}$ cannot depend on the remaining moduli z_i . Therefore $t_s \sim \epsilon^{1/2}$ all along $\Delta_s = 0$ and (3.5) holds⁷.

We can give now also an explanation for the overall shift of $\frac{1}{2}$, which we had to take into account to extract the spectrum from the coefficient r_s of F_1^{top} , as the contribution of the Jacobian of the mirror map. Namely, at $t_s = 0$, not only $z_s = 0$, but also its first derivative:

$$\frac{\partial z_s}{\partial t_s} \sim t \sim (z_s - \frac{1}{4})^{\frac{1}{2}} \sim \Delta_s^{\frac{1}{2}} . \quad (3.8)$$

The differential operator \mathcal{L} is simply the restriction of one operator of the complete Picard–Fuchs system to the locus $z_i = 0$. In the $n = 2$ case it is not difficult to understand (3.5) as a consequence of the fact, that the z_s dependence of the period t_s is in general described by a series of hypergeometric functions, not only for $z_i = 0$. This follows from the structure of the generators of the Mori cone which are essentially of the form

$$\begin{aligned} l_s &= (1, 1, -2, 0, \dots, 0) , \\ \bar{l}_s &= (0, 0, 1, \dots) , \\ l_k &= (0, 0, 0, \dots) . \end{aligned} \quad (3.9)$$

where the l_s, \bar{l}_s, l_k correspond to the complex structure moduli z_s, \bar{z}_s, z_k . There can be modifications of the precise expression for \bar{l}_s , the important point being that there is a single vector \bar{l}_s which overlaps with the vector l_s of (3.6). If there would be more relations involving the vertices ν_i^* , $i = 0..2$, of the \mathbb{Z}_2 singularity, it would mean that this singularity is part of a more complicated singularity. In this case we can still switch off the perturbations corresponding to the vertices ν_j^* involved in the additional relations to get a situation as in (3.9); in fact we will use such a procedure below to generalize from the \mathbb{Z}_2 to the \mathbb{Z}_n case.

⁷ Actually it is important for this kind of argument that the topology of the singular locus Δ_s does not change when the remaining moduli are switched on. This happens in the case of the conifold discriminant Δ_c in (1.3).

From (3.9) and the expression for the fundamental period in the large complex structure limit [13], it follows that the z_s dependence can be described in terms of hypergeometric functions:

$$\omega_0 = \sum_{\bar{n}, n_1, \dots, n_k} F(\bar{n}, n_1, \dots, n_k) \bar{z}^{\bar{n}} z_1^{n_1} \dots z_k^{n_k} u_{\bar{n}}(z_s), \quad (3.10)$$

where $u_\mu(z_s) = (z_s)^{-\mu/2} {}_2F_1(-\frac{1}{2}\mu, -\frac{1}{2}\mu + \frac{1}{2}, 1; 1/z_s)$ for the precise choice of \bar{l}_s in (3.9). This kind of representation of the fundamental period has been introduced in [12], to determine the full period vector and the mirror maps for the two K_3 fibrations $(\mathbb{P}_{1,1,2,2,2}^4 \parallel 8)_{-168}^{2,86}$ and $(\mathbb{P}_{1,1,2,2,6}^4 \parallel 12)_{-252}^{2,128}$. For the discussion of the strong coupling singularity we focus on the z_s dependence which is more general; in the following we will use heavily the results and notations of ref. [12]. In fact the only information we need is that the z_s dependence of *all* the periods related to the special coordinates, is described by two series of function, namely u_n and the second linear independent solution to the hypergeometric equation, called v_n in [12]. From the transformation behavior of t_s under the local monodromy it follows that the z_s dependence of t_s is proportional to $v_n = u_n v_0 - (4z_s)^{-\frac{1}{2}} \sqrt{1-4z_s} f_n(\frac{1}{z_s})$, where f_n is a polynomial and $v_0 = 2 \ln((4z_s)^{-\frac{1}{2}}(1 + \sqrt{1-4z_s})) - i\pi$. The term proportional to v_0 gives rise to the limiting behavior (3.7), while the second term controls the part which depends on the remaining moduli:

$$t_s = -\frac{1}{\pi i} \ln((4z_s)^{-\frac{1}{2}}(1 + \sqrt{1-4z_s})) + z_s^{-\frac{1}{2}} \sqrt{1-4z_s} F(z_s, z_k). \quad (3.11)$$

Inversion yields $z_s = (q_s - 1)^2 f(q_s, q_1)$.

To generalize to \mathbb{Z}_n we use first the fact that each triple of successive vertices ν_{i-1}^* , ν_i^* , ν_{i+1}^* fulfils the same relation as the vertices in (3.4) and can be used to reach a limiting point described by the mirror map (3.7). This corresponds to the possibility to blow down each single \mathbb{P}_i^1 of the resolution while keeping the remaining volumes arbitrarily large.

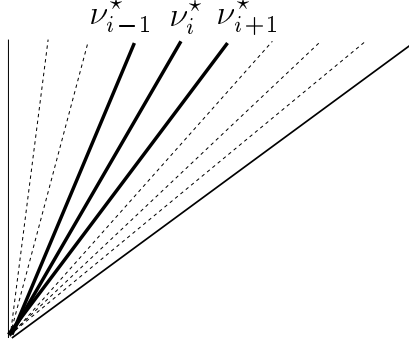


Fig.2 Three vertices of the blow up of the \mathbb{Z}_n singularity defining a limit where one exceptional \mathbb{P}_i^1 is blown down to a point.

In this way we obtain $n - 1$ reflections $\sigma_i : t_s^i \rightarrow -t_s^i$ acting on the special coordinates t_s^i . The global properties are governed by the intersection form Ω of the exceptional divisors E_i of complex codimension one, which are \mathbb{P}^1 bundles over the singular curve C fixed by the \mathbb{Z}_n action. The quadratic intersection form of the E_i defined by $J \cdot E_i \cdot E_j$, where J is the Kähler form of the ambient space, descends from the intersection form of the rational curves and is given by [28][18]:

$$J \cdot E_j^2 = -2\Lambda, \quad J \cdot E_i \cdot E_j = \Lambda, \quad |i - j| = 1, \quad J \cdot E_i \cdot E_j = 0, \quad \text{otherwise}, \quad (3.12)$$

where Λ is a numerical factor which does not matter in the present context. The full monodromy group of the \mathbb{Z}_n discriminant factor is generated by the reflections σ_i and has to leave invariant the intersection form Ω which is proportional to the Cartan matrix of A_{n-1} in virtue of (3.12). What we have described above are precisely the properties of the Weyl group of $SU(n)$! In fact the Weyl transformations can be recovered from the reflections and the products at the intersections by induction in n .

A nice illustration of how the gravitational index manages to fit the enhancement at the intersection of two $SU(2)$ factors is given by the second and third three moduli examples of table 4. The \mathbb{Z}_3 discriminant factor generalizing $\Delta_s = 1 - 4z_s$ for \mathbb{Z}_2 is

$$\Delta_s = 1 - 4z_s^1 - 4z_s^2 + 18z_s^1 z_s^2 - 27(z_s^1 z_s^2)^2. \quad (3.13)$$

The differential equation governing the behavior of the mirror map relating the special coordinates t_s^i to the algebraic moduli z_s^i is determined by two vectors of the Mori cone

$$l_s^1 = (0, 1, 1, -2, 0, \dots, 0), \quad l_s^2 = (1, 0, -2, 1, 0, \dots, 0) \quad (3.14)$$

and the solution of the corresponding differential equation at the limiting $SU(2)$ points $z_s^i = 0$, $z_s^j = \frac{1}{4}$ and the intersection point $z_s^1 = z_s^2 = \frac{1}{3}$ are given by (3.7) and

$$t_1 = t_2 = \ln \left(\frac{1 - z_s - \sqrt{1 - 2z_s - 3z_s^2}}{2z_s} \right), \quad z_s \equiv z_s^1 = z_s^2, \quad (3.15)$$

respectively.

On the $SU(2)$ lines the situation parallels the case of the \mathbb{Z}_2 singularities described above. At the intersection point $z_s^1 = z_s^2 = 1/3$ we expect an additional number of massless hyper multiplets fitting in representations of $SU(3)$. At this points the discriminant factor (3.13) is singular and becomes proportional to ϵ^2 instead of ϵ , where $\epsilon \sim z_s - 1$. Similarly the Jacobian contains a factor of ϵ instead of $\epsilon^{\frac{1}{2}}$, due to the fact that two of the three rows are $\sim \epsilon^{\frac{1}{2}}$. Finally in order to get the correct normalization for $SU(3)$ we have to rescale by a factor of 3 instead of 2. The resulting formula $b_s = (2r_s + 1) \times 3$ matches precisely the number of rational curves lying in the exceptional divisors of the blow down, shown in table 4. The number of hyper multiplets fits precisely into adjoint representations of $SU(3)$.

In view of the universal contribution of $\frac{1}{2}$ from the Jacobian for the case of $SU(2)$ and the jump at the intersection point of two $SU(2)$ factors it is suggestive to think about the opposite contributions of the Jacobian and the discriminant factors to the gravitational index as the opposite contribution of massless vector and hyper multiplets. Such an interpretation is supported by the by the cancellation appearing in the first example of table 4, where we find $b_s = (-1/2 + 1/2) \times 2 = 0$ for the discriminant factor $\Delta_s = (1 - 4z_s)$. The spectrum fitting these data is just an adjoint vector and hyper multiplet without additional matter. This spectrum can be further confirmed by calculating the monodromy and the periods in the local coordinates vanishing at Δ_s , where one finds that the two characteristic periods with eigenvalue minus one show no logarithmic behavior.

N_0	CY manifold	$j : \quad r_j \quad b_j$	$i : \quad s_i \quad c_i^2$	$n_{i,0,0}^{(0)}$	$n_{0,i,0}^{(0)}$	$n_{0,0,i}^{(0)}$	$n_{i,j,k}^{(1)}$
9	$(\mathbb{P}_{1,1,2,8,12}^4 \parallel 24)_{-480}^{3,243}$	$c_1 : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{3}{6} \quad -\frac{2 \cdot 0}{6}$ $c_2 : -\frac{1}{6} \quad -\frac{1}{6}$	$1 : -\frac{52}{6} \quad 92$ $2 : -3 \quad 24$ $3 : -5 \quad 48$	480	0	-2	$n_{i,j,k}^{(1)} = 0,$ $i < 2$
10	$(\mathbb{P}_{1,2,3,3,3}^4 \parallel 12)_{-132}^{3,69}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{5}{6} \quad -\frac{2 \cdot 2}{6}$	$1 : -5 \quad 48$ $2 : -\frac{17}{3} \quad 56$ $3 : -3 \quad 24$	56	4	4	$n_{i,j,k}^{(1)} = 0,$ $i < 2$
11	$(\mathbb{P}_{1,2,3,3,9}^4 \parallel 18)_{-192}^{3,99}$	$c : -\frac{1}{6} \quad -\frac{1}{6}$ $s : -\frac{4}{6} \quad -\frac{2 \cdot 1}{6}$	$1 : -\frac{27}{6} \quad 42$ $2 : -3 \quad 24$ $3 : -\frac{16}{3} \quad 52$	252	2	2	$n_{i,j,k}^{(1)} = 0,$ $i < 2$
				42	0	0	
				84862	0	0	

Table 4: *Three moduli K_3 -fibration Calabi-Yau manifolds. The first has a Z_2 singular curve with an exceptional Z_4 point. The last two have Z_3 singular curves.*

4. Comments

We have analyzed two types of transitions between topological different Calabi-Yau manifolds, which have a physical interpretation in terms of new branches in the moduli space arising from additional massless degrees of freedom. The first describes the inverse transition of the conifold transition described in [4]. The second one is related to curve singularities on the Calabi-Yau manifold and corresponds to a situation with enhanced non-abelian gauge symmetries with a non asymptotically free spectrum. They have in common the fact that on the transition point the volume of certain rational curves becomes zero. In the examples where a dual heterotic description exists, the singularities lie in the strongly coupled regime. This provides a new, stringy realization of non-abelian gauge symmetries through solitonic degrees of freedom in a $N = 2$ theory at strong string coupling, invisible in the perturbation theory.

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